Optimizing Polar Codes for Noisy Communication Systems: A Study on Code Rate and Error Correction Trade-Offs

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Abstract-*In this paper, we present an in-depth analysis of the performance of polar codes when applied to communication systems operating under additive white Gaussian noise (AWGN) conditions. Polar codes, known for their capacity to achieve channel capacity with low decoding complexity, are evaluated in terms of their bit error rate (BER) and frame error rate (FER) performance. A key focus of the study is the investigation of different code rates, defined as* $R = K/N$, *where* K *represents the number of information bits and* N *is the total code length. By varying the code rates, we explore the trade-offs between coding efficiency and error correction performance, providing practical insights into how to optimize polar codes for use in noisy communication environments. The construction of polar codes in this study is based on the Bhattacharyya method, a well-established technique for designing reliable codes by leveraging the channel's error probabilities. This approach ensures that the most reliable bit channels are prioritized, thereby enhancing the error-correcting capability of the codes. Additionally, we employed a successive cancellation decoder (SCD), which is an efficient decoding algorithm specifically designed for polar codes. To make the Bhattacharyya algorithm more comprehensible, we present its workflow in the form of a detailed flowchart, highlighting each step in the code construction process. Through extensive simulations, we evaluated the BER and FER for various code rates and block lengths. Our results reveal that the performance of polar codes improves significantly with an increase in block size N. This observation highlights the critical role of redundancy bits, which contribute to better protection of the transmitted signal against noise-induced errors. At the same time, our findings demonstrate the inherent trade-off, as higher redundancy implies a lower information rate. This paper emphasizes the importance of optimizing polar code parameters to achieve a balance between efficiency and robustness, making them highly suitable for practical applications in modern communication systems. This study provides valuable guidance for researchers and engineers working on enhancing error correction techniques in noisy channels.*

Keywords:*Polar Codes, AWGN, Bhattacharyya-parameters, Code Length, Satellite-Communications.*

INTRODUCTION

Digital transmission systems transmit information between a transmitter and a receiver using a medium such as propagation over a channel. The signals transmitted may be either directly digital in origin, such as data networks, or analogue in origin (voice, image...) but converted into digital form. The primary task of the transmission system is to carry the information from the source to the receiver as reliably as possible. Generally, in digital communications, the alphabet used is binary. This type of data transmission is particularly interesting because it allows the use of many techniques such as encryption of information and error-correcting codes, the latter being the focus of extensive current research.

Error-correcting codes, also known as channel encoding, have become increasingly essential in digital transmission systems over microwave channel. Advances in technology have led to the introduction of a coding function, which only moderately increase in the complexity of the equipment while providing significant benefits in terms of reliability and performance. Channel coding adds redundancy to the transmitted information, enabling the detection and correction of errors that occur during transmission. Channel coding adds redundancy to the transmitted information, enabling the detection and correction of errors that occur during transmission.

In 1948, C. Shannon published the founding article of information theory (Shannon, 1948), which formalized the fundamental concepts of information transmission mathematically. In particular, he introduced the concept of the capacity of a noisy channel, which now bears his name, as the maximum theoretical information flow in the presence of noise. This notion finds its relevance in its famous “second theorem”, called channel coding theorem, which states that by adding redundancy to the initial message, it is possible to transmit with an arbitrarily low probability of error, for any yield below channel capacity. Let’s suppose that we send a binary information message and that the detection is done bit by bit in the receiver. If the message sent does not contain redundancy, each bit sent is essential and any transmission error leads to an irreversible loss of information. On the other hand, if redundant bits are introduced in the message, it may be possible to detect and even correct transmission errors. For this reason, we use error-correcting codes.

Several researches are devoted in this field, such as the convolutional codes which were invented by Elias in (Elias, 1955). In 1993, Berrou described parallel concatenation which gave rise to the birth of turbo codes (Berrou *et al*., 1993; Stephane *et al.*, 1994); and to the resurgence of LDPC codes, following the work of Mckay in (MacKay *et al.*, 1996), then those of Richardson et al. who introduced the density evolution technique for the analysis and optimization of these codes (Chung *et al.*, 2001). However, despite this considerable progress, the question of the attainability of the Shannon limit continued to resist researchers. In 2009, Arikan provided the solution through polar codes (Arikan, 2009).

Polar codes are a revolutionary development in the field of error-correcting codes. They are the first class of codes with a proven capacity to achieve the Shannon capacity for a wide range of channels, including the Additive White Gaussian Noise (AWGN) channel.

Polar codes are constructed using a process known as channel polarization, which transforms a set of identical channels into a set of polarized channels, some of which are highly reliable and others highly unreliable. By transmitting information over the most reliable channels, polar codes ensure robust error correction.

This study investigates the performance of polar codes over an AWGN channel, utilizing the Bhattacharyya construction method and a successive cancellation decoder (SCD) for decoding.

- Experiments were conducted for various values of N in the AWGN channel.
- The influence of the code rate $R = K/N$ on performance was examined.

This paper is organized as follows:

In section II, we explain the basic concepts of polar codes. Then in section III, we propose the construction algorithm for an AWGN channel and a scheme for our transmission system. In section IV, we show the simulation results, and we finally conclude in section V.

POLAR CODES

Encoding

A Polar Code is specified by $(N, K, \mathcal{F}, v_{\mathcal{F}})$, where N is the length of the code in bits, K is the number of information bits, $v_{\mathcal{F}}$ is the binary vector of length $N - K$ (frozen bits) and $\mathcal{F}$ is a subset of $N - K$ indices of $\{0,1,\dots,N-1\}$ i.e. the location of frozen bits.

The encoding operation is described below.

Let $n \triangleq log2(N)$ and$G \triangleq F^{\otimes n} = F \otimes \dots \otimes F$ be the n-fold Kronecker product of $F \triangleq \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$; where $Gis$the generating matrix.

The code word X can be calculated as follows:

$$X = U \times G = U \times F^{\otimes n} \quad (1)$$

Figure 1 shows the generator matrix and the coding graph of a polar code of size $N = 2$ and message $U = [u_1 u_2]$.

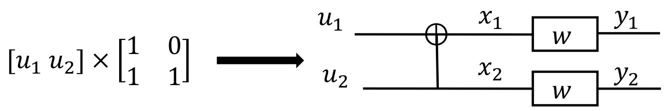

Figure 1: Polar Encoder with DMC for N=2

Figure 2 shows the coding graph of a polar code $(8,4)$, where the code rate is $R = 1/2$. This indicates that half of the bits in the vector$U$ are frozen bits, while the remaining half are information bits, represented as$[u_1 u_2 u_3 u_4]$. Note that the frozen bits are fixed to zero, as described in (Arikan, 2009).

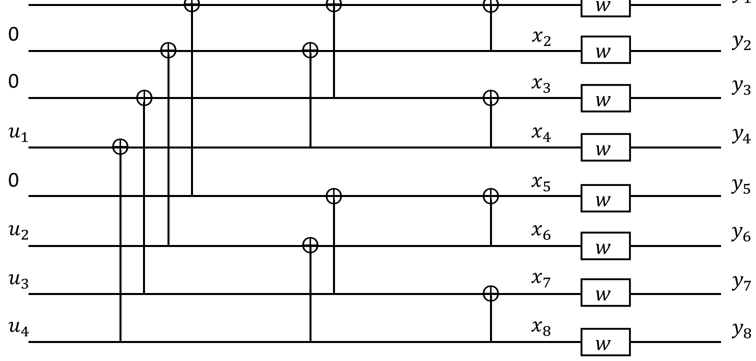

Figure 2: Polar Encoder with DMC for N=8

Channel Polarization

Channel polarization, proposed by Arikan (Arikan, 2009), is a technique for designing code sequences that achieve the symmetric capacity $I(W)$ of any binary input discrete memoryless channel (B-DMC). This method transforms N independent and identical copies of a binary channel W through a series of combining and splitting operations. The transformation generates a new set of binary input channels $W(i)$, whose capacities asymptotically approach 0 or 1, as thoroughly described in Arikan's groundbreaking study (Arikan, 2009).

Construction: The selection of the set $\mathcal{F}$ is a critical aspect of polar code construction. This set is determined using an algorithm introduced in (Arikan, 2008; Mori *et al.*, 2009), which relies on the Bhattacharyya parameters. These parameters, denoted as $Z(W)$, serve as a measure of the reliability of a channel W and are defined as follows:

$$Z(W) = \int \sqrt{W(y|0)W(y|1)}\, dy \qquad (2)$$

There are three types of recursive Bhattacharyya parameter relations. Existing research (Peng Shi et al., 2012; Shengmei Zhao et al., 2011) indicates that Type 1 exhibits the highest performance among them.Figure 3 illustrates the polarization effect for the case where W is a AWGN channel with $E_b/N_0 = 1.5_{dB}$.

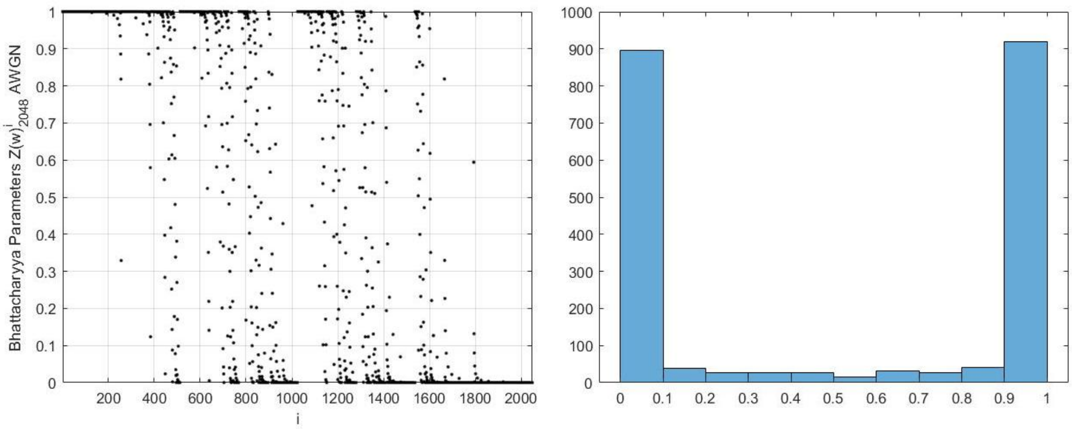

Figure 3: Bhattacharyya Parameters for $N = 2048$ with $R = 1/2$ and $E_b/N_0 = 1.5_{dB}$ for AWGN Channel

Successive Cancellation Decoder (SCD)

The successive cancellation decoding (SCD) algorithm operates using the same coding scheme illustrated in Figure 3, but with the processing direction reversed. In this approach, likelihoods are computed based on the channel observations at the right end and propagated toward the left. Once a likelihood reaches the left end, a decision is made, and this decision is propagated back from left to right, updating the bits throughout the graph, as detailed in (Arikan, 2009) and (Vangala *et al.*, 2014). Notably, reversing the processing direction can sometimes provide performance benefits, making this method particularly intriguing.

POLAR CODES WITH AWGN CHANNEL

We will now present the block diagram of a digital transmission system with a binary source. Figure 4 shows the different basic modules of this communication chain:

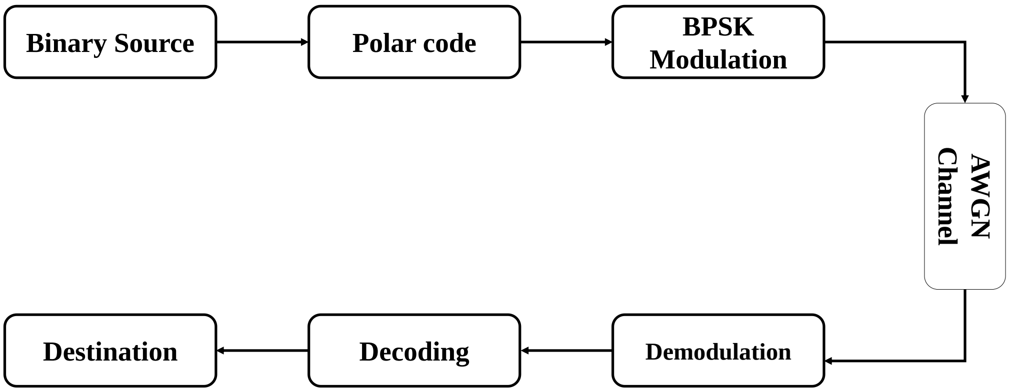

Figure 4: Diagram of a Digital Communication System

Polar Code Construction for an AWGN Channel

The AWGN channel has a discrete input x_k with a value in {0,1} in the case of a binary channel and a continuous output y_k (BI-AWGN: Binary Input-Additive White Gaussian Noise Channel). This channel is a continuous stationary channel, meaning it does not depend on time and lacks memory effect, y_k depends solely on x_k. The Gaussian channel is characterised by the following relationship between input x_k and output y_k :

$$y_k = x_k + n \quad (3)$$

The channel noise n is a random variable that adds to the transmitted modulated signal. This variable n follows a Gaussian distribution, with mean zero and variance $\sigma^2 = {N_0}/{2}$ where N_0 is the power spectral density of the noise. Therefore, this variance σ is as a function of the Signal-to-Noise Ratio (SNR) denoted as:

$$SNR = \frac{R.E_b}{N_0} = \frac{1}{2\sigma^2} \quad (4)$$

This equation highlights the relationship between the variance of the Gaussian noise and the SNR, which is crucial in understanding the performance of communication systems in the presence of noise.

Where E_b is the energy emitted per bit. Figure 5 shows a schematic of an AWGN channel.

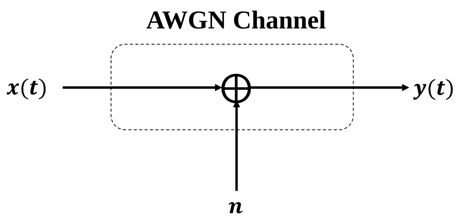

Figure 5: AWGN Channel with Mean Zero and Variance σ^2

Now we will calculate the initial Bhattacharyya parameter $Z(W_1^1)$ for a BI-AWGN channel. The Bhattacharyya parameter for a continuous channel can be calculated as follows (Li *et al.*, 2013):

$$z(w) = \int_{-\infty}^{+\infty} \sqrt{W(y|0)W(y|1)}\, dy \quad (5)$$

Thus, we employ a BPSK, it is the simplest form of PSK. This modulation is the most robust of all PSKs because it takes a large amount of signal distortion for the demodulator to misjudge the received symbol. The $W(y|0)$ and the $W(y|1)$ can be calculated as below:

$$W(y|0) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y+1)^2}{2\sigma^2}} \quad (6)$$

$$W(y|1) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-1)^2}{2\sigma^2}} \quad (7)$$

From (6) and (7), $Z(W_1^1)$ for the AWGN channel is:

$$Z(W_1^1) = e^{-\frac{RE_b}{N0}} \quad (8)$$

We will now discuss our polar code construction algorithm as shown in figure 6. Initially, we will initialize the number of information bits K, the codeword size N, and the energy per bit to noise power spectral density ratio in dB E_b/N_0 .Next, we will calculate the initial Bhattacharyya parameter for an AWGN channel using the previous formula (8). The Bhattacharyya parameter can be considered as the maximum value of the transmission error probability. The algorithm proceeds with two loops to compute the sequence of Bhattacharyya parameters for the split channels. The first loop increments each time until it reaches the size of the code word N. The second loop increments the index of the split channels each time until a certain condition is verified. At the end of this algorithm, the Bhattacharyya parameters are displayed in the form of a vector in descending order. In this vector, the first element corresponds to largest value of the error probability, while the last element corresponds to the smallest value of the error probability.

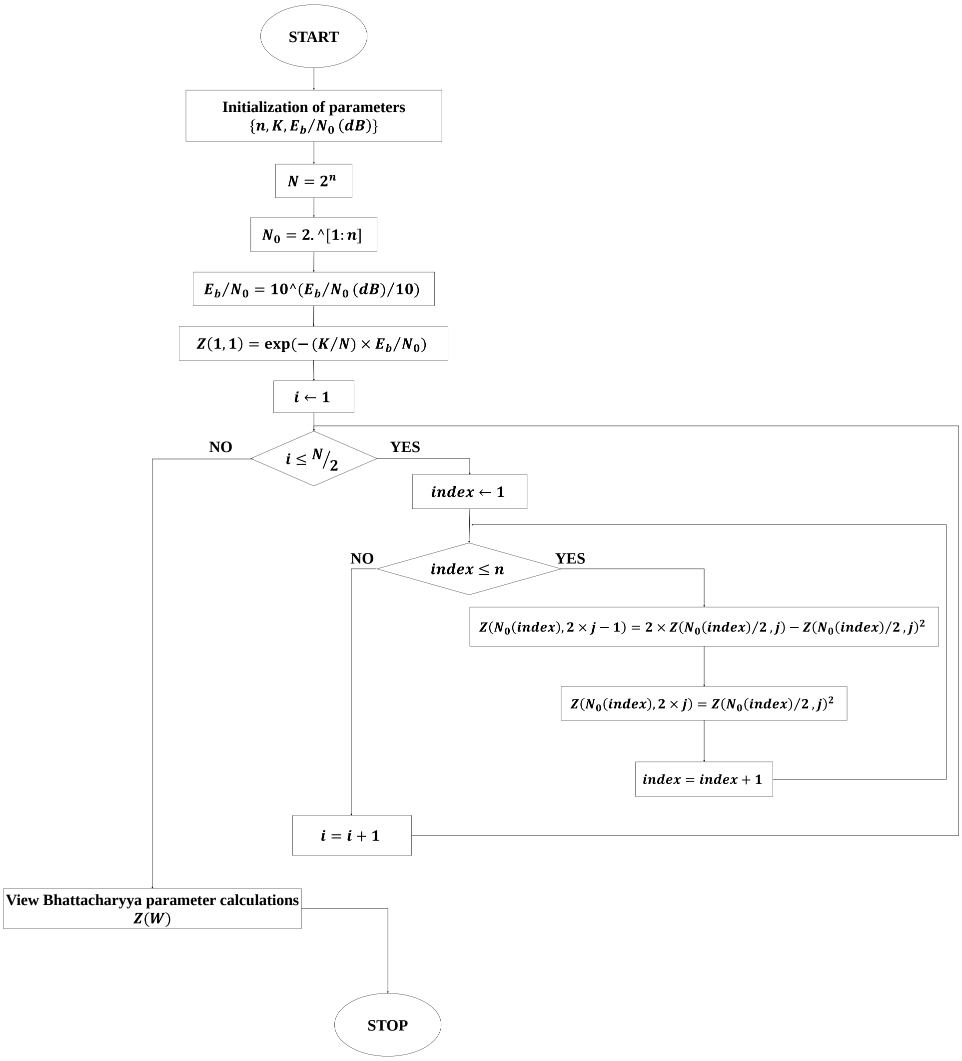

Figure 6: Bhattacharyya Algorithm in Flowchart Form

RESULTS

In this section, we will present the performance of polar codes with different code lengths N and the effect of the rate R on these codes, using an SCD decoder.

Figure 7, this figure shows the importance of the block length N for a polar code with $R = 1/2$. It demonstrates that as N increases, both the frame error rate (FER) and bit error rate (BER) decrease. From this, we conclude that polar codes perform better with larger block sizes N.

Figure 8, in this figure, we fix the block size at $N = 2048$ and observe that smaller R ratios result in better performance of polar codes. This indicates the importance of redundancy bits in protecting the signal from transmission errors.

Figure 9, here, we focus on the information bits. If the message does not contain a sufficient number of redundant bits, each transmitted bit becomes essential, and any transmission error leads to an irreversible loss of information. From this, we conclude that a higher R increases the probability of error.

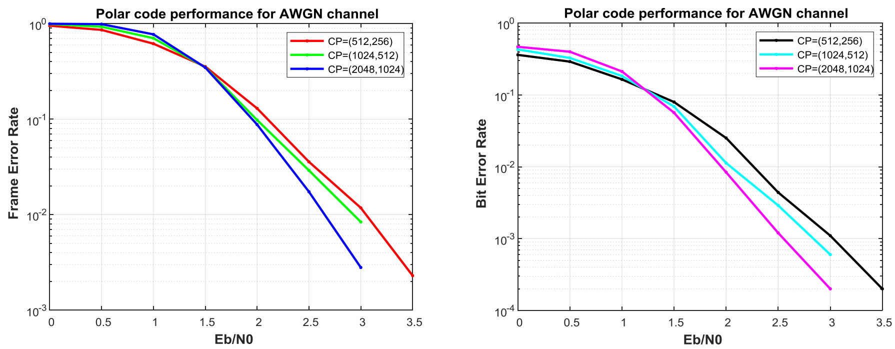

Figure 7: Polar Code Performance for AWGN Channel with N=512,1024,2048 and R=1/2

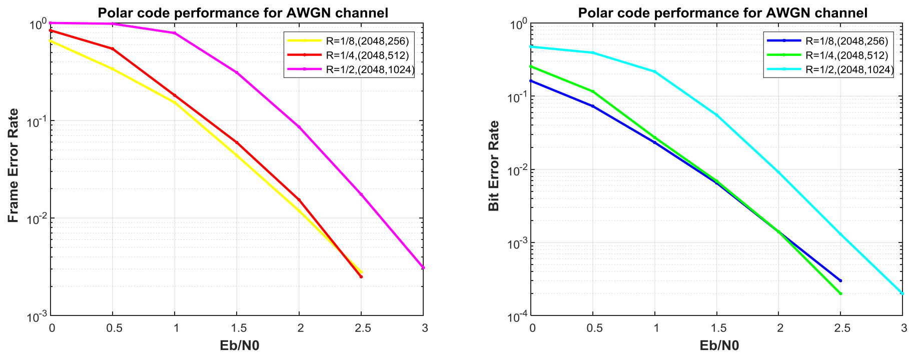

Figure 8: Polar Code Performance for AWGN Channel with N=2048 and R=1/2, 1/4, 1/8

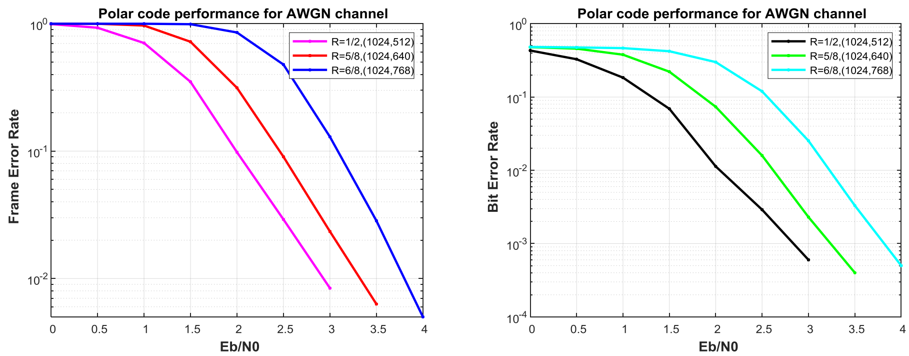

Figure 9: Polar Code Performance for AWGN Channel with N=1024 and R=1/2, 5/8, 6/8

CONCLUSION

In this paper, we conducted a complete performance analysis of polar codes in the AWGN channel, with a focus on bit error rate (BER) and frame error rate (FER) across various code rates. Our simulations highlighted the effect of different code rates on error correction performance, providing important insights into the trade-offs between code rate and error correction efficiency. These findings help to optimize polar code design for practical communication systems by directing code rate selection for specific performance objectives. Future research directions could explore the performance of polar codes under more complex and realistic channel models, such as fading or burst error channels. Additionally, investigating advanced decoding techniques, including machine learning-assisted algorithms or hybrid decoding strategies, could further enhance error correction efficiency. Expanding the scope to include multi-user communication scenarios and the integration of polar codes in next-generation communication standards, such as 6G, also presents promising areas for further study.

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